

Presymplectic current and the inverse problem of the calculus of variations

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Abstract

The inverse problem of the calculus of variations asks whether a given system of partial differential equations (PDEs) admits a variational formulation. We show that the existence of a presymplectic form in the variational bicomplex, when horizontally closed on solutions, allows us to construct a variational formulation for a subsystem of the given PDE. No constraints on the differential order or number of dependent or independent variables are assumed. The proof follows a recent observation of Bridges, Hydon and Lawson and generalizes an older result of Henneaux from ordinary differential equations (ODEs) to PDEs. Uniqueness of the variational formulation is also discussed.

1 Introduction

Many systems of partial differential equations (PDEs for short) that appear in physics are variational. That is, they are equivalent to Euler-Lagrange (EL) equations of some local Lagrangian density. The question of whether some arbitrarily given PDE is variational is known as the *inverse problem of the calculus of variations*. It has attracted a significant amount of attention in the past [2, 21]. In physics, a variational formulations endows the algebra functions on the phase space of a classical theory with Poisson structure, which in turn determines the corresponding commutation relations in quantum theory. One important consequence of the inverse problem is the ability to reconstruct a variational principle and hence the quantum commutation rules (whether unique or not) directly from the equations of motion [24, 15].

This problem comes in two versions, the harder *multiplier* problem and the easier *non-multiplier* problem. The idea behind the solution of the non-multiplier problem for ordinary differential equations (ODEs) was known already to Helmholtz and the necessary and sufficient conditions for a positive

solution are known as the *Helmholtz conditions*. The generalization of these conditions to PDEs is elegantly stated in terms of the variational bicomplex [3, 4]. The multiplier problem is much less understood, with significant results obtained for ODEs by Douglas [12] and for PDEs by Anderson and Duchamp [5]. A criterion certifying a positive solution in the case of second order ODEs was given later by Henneaux [15], which was later extended to higher orders [17, 1]. This criterion is essentially the existence of a (pre)symplectic form on the space of dependent variables that is conserved by the flow of the ODE.

The work of Henneaux proved difficult to generalize to partial differential equations (PDEs). For one thing, it is not immediately clear what is the right analog of the symplectic form. Also, Henneaux represented a second order ODE as a vector field on the tangent bundle of the configuration space. Finally, the conservation condition was expressed as annihilation by the Lie derivative of the ODE vector field (with suitable generalization in the time dependent case). However, we can now say that Henneaux's argument can in fact be (at least partially) generalized to the PDE case. The analog of the symplectic form is the covariant presymplectic current (Section 3). And the analog of the geometric vector field representation of an ODE is the geometric representation of a PDE as a submanifold of the jet bundle of the dependent variables (Section 2). Within this context, the corresponding generalization of Henneaux's criterion certifying a positive solution of the multiplier inverse problem for PDEs was given only very recently in [9, 16]. This key observation appeared as a side remark in that work, with only a sketch of the details and without placing it within the context of the inverse problem. Unfortunately, this solution of the inverse problem is only partial, as the Lagrangian density produced by the proposed procedure (Section 4) produces a variational system that need not be equivalent to the original PDE system, though this variational system will admit all solutions of the original PDE system among its own solutions.

This note aims at highlighting and clarifying the above result by placing it in the appropriate geometric context. Section 2 introduces some basic background on the geometric formulation of PDEs using jet bundles and the associated variational bicomplex. Section 3 shows how a conserved presymplectic current density arises in variational problems. Conversely, Section 4 shows that the presence of a conserved presymplectic current density is equivalent to the fact that the solutions of the given PDE system are also solutions (but perhaps not the only ones) of a variational PDE system. Section 5 discusses the non-uniqueness of the Lagrangian constructed in Section 4. Finally, conclusions are in Section 6.

2 Jet bundles and PDE systems

This section briefly defines the basic notions needed for the geometric formulation of PDE systems in terms of jet bundles, including associated geometric structures, and fixes some notation. For a detailed discussion of jet bundles see [20, 18]. The variational bicomplex is discussed in detail in [3, 4].

Let $F \rightarrow M$ be a vector bundle over the base space M , an n -dimensional smooth manifold. Since we will be only interested in local questions, assume that both F and M are topologically trivial (contractible). The arguments in this paper can be straightforwardly generalized to smooth bundles with non-trivial fiber and base space topologies [13]. Let $\Gamma(F)$ denote the space of smooth sections of F . Denote by T^*N the cotangent bundle for any smooth manifold N and by $\Lambda^k N = \bigwedge_M^k T^*N$ the bundle of alternating k -forms over N . Denote by $\Omega^k(N) = \Gamma(\Lambda^k N)$ the space of differential forms on N , by d the corresponding de Rham differential and by $\Omega^*(N)$ the total complex of differential forms graded by degree.

The k -jet $j_x^k \phi$ of a section $\phi \in \Gamma(F)$ at $x \in M$ can be defined as the equivalence class of sections $\psi \in \Gamma(F)$ that have coinciding Taylor polynomials up to and including order k at point x in any (bundle adapted) local coordinate system. Thus, k -jets are a coordinate invariant way of capturing the derivatives of a section up to order k . Local coordinates (x^i, u^a) on F naturally extend to local coordinates (x^i, u_I^a) , $|I| \leq k$ on $J^k F$, where I with $|I| = l$ is a multi-index $(i_1 i_2 \cdots i_l)$. The coordinates of the k -jet of ϕ at x are

$$(x^i, u^a, u_i^a, \dots, u_I^a)(j_x^k \phi) = (x^i, \phi^a, \partial_i \phi^a, \dots, \partial_I \phi^a)(x), \quad |I| = k, \quad (1)$$

where we have used the shorthand $\partial_I = \partial_{i_1} \cdots \partial_{i_k}$. These k -jets form the space $J_x^k F$, which is a fiber of the vector bundle $J^k F \rightarrow M$ of k -jets over M . The 0-jets are identical with the underlying bundle, $J^0 F \cong F$. Jet bundles come with natural projections $J^k F \rightarrow J^l F$, for any $k \geq l$, which simply discard the information about all derivatives higher than l . This projection gives $J^k F$ the structure of an affine bundle over $J^l F$. The projective limit

$$J^\infty F = \varprojlim_k J^k F \rightarrow \cdots \rightarrow J^2 F \rightarrow J^1 F \rightarrow J^0 F \rightarrow M \quad (2)$$

is called the ∞ -jet bundle. A smooth function on $J^\infty F$ is the pullback of a smooth function on some $J^k F$ for some $k < \infty$. That is, a smooth function on $J^\infty F$ always depends only on finitely many components of an ∞ -jet given to it as argument.

A section $\phi \in \Gamma(F)$ naturally gives rise to the section $j^k \phi: M \rightarrow J^k F$, with $j^k \phi(x) = j_x^k \phi$, called the k -jet prolongation of ϕ . Similarly, a (not necessarily linear) bundle morphism $f: J^k F \rightarrow E$ (which determines a differential operator $f \circ j^k: \Gamma(F) \rightarrow \Gamma(E)$ of order k), naturally gives rise to a bundle morphism $p^l f: J^{k+l} F \rightarrow J^l E$, with $p^l f(j_x^{k+l} \phi) = j_x^l (f \circ j^k(\phi))$, called the l -prolongation of f .

In the geometric formulation, a PDE system on F of order k is a submanifold $\mathcal{E} \subset J^k F$ that satisfies the regularity conditions of being closed and that $\mathcal{E} \rightarrow M$ is a smooth sub-bundle of $J^k F \rightarrow M$. To connect with the usual notion of a PDE, we note that there always exists (at least up to possible global topological obstructions [14, §7]) a vector bundle $E \rightarrow M$ and a bundle morphism $f: J^k F \rightarrow E$ such that a section $\phi \in \Gamma(F)$ satisfies the system of differential equations $f \circ j^k(\phi) = 0$ iff the image of $j^k \phi \in \Gamma(J^k F)$ is contained in

\mathcal{E} , that is $j^k \phi$ is actually a section of $\mathcal{E} \rightarrow M$. We call the pair (f, E) an equation form of \mathcal{E} , which is in general not unique. The equation form can always be chosen to be regular, which means that any smooth function L on $J^k F$ vanishes on \mathcal{E} iff it can be written, in local coordinates (x^i, u^a) and (x^i, v^B) on F and E , as $L(x^i, u_I^a) = L_B(x^i, u_I^a) f^B(x^i, u_I^a)$, with smooth coefficients L_B . Conversely, any regular equation form (f, E) defines a PDE system \mathcal{E}_f given by the zero set of f . The l -prolongation $\iota_l: \mathcal{E}^l \subset J^{k+l} F$ is defined as the PDE system $\mathcal{E}_{p^l f}$ corresponding to the equation form $(p^l f, J^l E)$. The l -prolongation comes with a natural projection $p_l: \mathcal{E}^l \rightarrow \mathcal{E}$, which simply restricts the appropriate projection of jet bundles. Note that p_l need not be surjective if the PDE system has non-trivial integrability conditions. Below, we deal with ∞ -prolongations $\iota_\infty: \mathcal{E}^\infty \subset J^\infty F$ and $(p^\infty f, J^\infty E)$. We assume that both are regular.

The de Rham differential acting on $\Omega^*(J^\infty F)$ can be naturally written as the sum

$$d = d_h + d_v \quad (3)$$

of the, respectively, horizontal and vertical differentials. Each is nilpotent and they anticommute:

$$d_h^2 = d_v^2 = 0, \quad d_h d_v + d_v d_h = 0. \quad (4)$$

If d_M is the de Rham differential on M , $\phi \in \Gamma(F)$ is any section and $L \in \Omega^*(J^\infty F)$ any differential form, then the defining property of the horizontal differential is that it is intertwined with d_M via the pullback of differential forms along sections:

$$(j^\infty \phi)^* d_h L = d_M (j^\infty \phi)^* L. \quad (5)$$

Differential forms on $J^\infty F$ have the following natural subspaces: the purely horizontal forms $\Omega^{h,0}(F)$ that are generated by pullbacks of forms from $\Omega^h(M)$ along the natural projection $J^\infty F \rightarrow M$ and the purely vertical forms $\Omega^{0,v}(F) \subset \Omega^v(J^\infty F)$ that are annihilated by the pullback $(j^\infty \phi)^*$ of any section $\phi: M \rightarrow F$. Purely horizontal and purely vertical forms generate $\Omega^*(J^\infty F)$ as a graded algebra. The subspaces of homogeneous horizontal and vertical degrees are denoted by $\Omega^{h,v}(F) \subset \Omega^{h+v}(J^\infty F)$. The differentials d_h and d_v are then, respectively, of purely horizontal degree 1 and of purely vertical degree 1. In local coordinates (x^i, u_I^a) , horizontal forms are generated by $d_h x^i = dx^i$, while vertical forms are generated by $d_v u_I^a$. The total bi-differential bi-graded algebra $(\Omega^{*,*}(F), d_h, d_v)$ is called the variational bicomplex of the vector bundle $F \rightarrow M$. Within this bicomplex, we can define the horizontal, $H^{*,*}(d_h)$, and vertical, $H^{*,*}(d_v)$, cohomology groups in the obvious way.

The horizontal and vertical degrees, as well as differentials, survive restriction to the ∞ -prolonged PDE system $\iota_\infty: \mathcal{E}^\infty \subset J^\infty F$. Thus, the differential forms $\Omega^*(\mathcal{E}^\infty)$ can also be given the structure of a bi-differential bi-graded algebra $(\Omega_{\mathcal{E}}^{*,*}(F), d_h^{\mathcal{E}}, d_v^{\mathcal{E}})$, where $\Omega_{\mathcal{E}}^{*,*}(F) \cong \Omega^*(\mathcal{E}^\infty)$. The cohomology groups $H_{\mathcal{E}}^{*,*}(d_h)$ are called the characteristic cohomology groups of the PDE system \mathcal{E} . Characteristic cohomology classes can be identified with important geometric and algebraic properties of \mathcal{E} . For instance, a representative of an element of $H_{\mathcal{E}}^{n-1,0}(d_h)$ is a non-trivial conservation law of \mathcal{E} [7, 10, 23].

3 Covariant presymplectic current

With notation following Section 2, a *local action functional* of order k on F is a function $S[\phi]$ of sections $\phi \in \Gamma(F)$,

$$S[\phi] = \int_M (j^k \phi)^* \mathcal{L}, \quad (6)$$

where the *Lagrangian density* $\mathcal{L} \in \Omega^{n,0}(F)$ is a purely horizontal n -form on $J^k F$. It is local because, given a section ϕ and local coordinates (x^i, u_I^a) on $J^k F$, the pullback at $x \in M$ can be written as

$$((j^k \phi)^* \mathcal{L})(x) = \mathcal{L}(x^i, \partial_I \phi^a(x)), \quad (7)$$

which depends only on x and on the derivatives of ϕ at x up to order k . The integral over M will be considered formal, since all the necessary properties will be derived from \mathcal{L} . Incidentally the usual variational derivative of variational calculus can be put into direct correspondence with the vertical differential d_v on this complex, which is how the name *variational bicomplex* was established.

In Section 2 we introduced the variational bicomplex $(\Omega^{h,v}(F), d_h, d_v)$ of vertically and horizontally graded differential forms on $J^\infty F$. A Lagrangian density, being of top horizontal degree, is then a closed element of $\Omega^{n,0}(F)$. Usually, Lagrangian densities are considered equivalent if they differ by a horizontally exact term, for example \mathcal{L} and $\mathcal{L} + d_h \mathcal{B}$, where $d_h \mathcal{B}$ is often referred to as a *boundary term*. In other words, we should think of \mathcal{L} not just as an element of $\Omega^{n,0}(F)$, but rather a representative of a cohomology class $[\mathcal{L}] \in H^{n,0}(d_h)$.

Note that even though \mathcal{L} can be thought of as a form on $J^k F$, below we will carry out all calculations on $J^\infty F$, with the proviso that all intermediate formulas could have been projected onto jet bundles of some finite order, bounded throughout the calculation. When necessary, we shall make use of a local coordinate system (x^i, u_I^a) on $J^\infty F$.

Using integration by parts if necessary, we can always write the first vertical variation of the Lagrangian density as

$$d_v \mathcal{L} = \text{EL}_a \wedge d_v u^a - d_h \theta. \quad (8)$$

All terms proportional to $d_v u_I^a$, $|I| > 0$, have been absorbed into $d_h \theta$. In the course of the performing the integrations by parts, the coefficients EL_a can acquire dependence on jets up to order $2k$. The form θ is not uniquely specified, as one can freely substitute $\theta \rightarrow \theta + d_h \sigma$, where σ is any form¹ in $\Omega^{n-2,1}(F)$. Thus, the jet order of θ is not bounded from above (due to the arbitrariness in σ). However, as can be seen from integration by parts in local coordinates, θ can always be chosen to depend on jets of order no higher than $2k - 1$. The forms EL_a define a bundle morphism

$$\text{EL}: J^{2k} F \rightarrow \tilde{F}^* = \Lambda^n M \otimes_M F^*, \quad (9)$$

¹Actually, $d_h \sigma$ could also be replaced by any merely *closed* form. However, the cohomology groups $H^{*,v}(F)$ for $v > 0$ are always trivial [3, Ch.5], so there is no loss in generality.

where $F^* \rightarrow M$ is the dual vector bundle to $F \rightarrow M$. The equation form (EL, \tilde{F}^*) determines the so-called *Euler-Lagrange (EL) PDE system* $\iota: \mathcal{E}_{\text{EL}} \subset J^{2k}F$ associated with the Lagrangian density \mathcal{L} or equivalently the local action functional $S[\phi]$.

A PDE system with an equation form given by Euler-Lagrange equations of a Lagrangian density is said to be *variational*. The form $\theta \in \Omega^{n-1,1}(F)$ is referred to as the *presymplectic potential current density*. Applying the vertical differential to θ we obtain the *presymplectic current density* (or the *presymplectic current density defined by \mathcal{L}* if the extra precision is necessary).

$$\omega = d_v \theta, \quad (10)$$

with $\omega \in \Omega^{n-1,2}(F)$. Since the vertical differential does not increase the jet order, the jet order of ω is bounded by that of θ . Note that changing \mathcal{L} by a boundary term $d_h \mathcal{B}$ changes θ by the vertically exact term $-d_v \mathcal{B}$. Hence, ω is not altered by this change. If one changes θ by the addition of a horizontally exact term $d_h \sigma$, then the presymplectic current density also changes by a horizontally exact term, $\omega \rightarrow \omega - d_h d_v \sigma$.

The terminology for ω comes from classical field theory in physics. Restricted to solution of \mathcal{E}_{EL} and integrated over a closed codim-1 surface in M (most often a Cauchy surface) ω defines a presymplectic form on the space of solutions of the EL equations. This (possibly infinite dimensional) space, known as the phase space, is then a presymplectic manifold. If the initial value problem on this surface of integration is well posed, it is even a symplectic manifold. This method of construction the symplectic form on the phase space of a classical field theory is known as the *covariant phase space method* [25, 6, 11, 19, 8].

Since ω is, like Λ , to be integrated over a boundaryless submanifold, we only care about its equivalence class $[\omega] \in \Omega^{n-1,2}(F)/d_h \Omega^{n-2,2}(F)$. Thus, the discussion above has shown the following

Proposition 1. *Given a Lagrangian density $\mathcal{L} \in \Omega^{n,0}(F)$ of order k , there exist forms $\theta \in \Omega^{n-1,1}(F)$ and $\omega \in \Omega^{n-1,2}(F)$, of jet order at most $2k-1$, as well as a section $\text{EL}: J^{2k} \rightarrow \tilde{F}^*$ that satisfy the equations*

$$d_v \mathcal{L} = \text{EL}_a \wedge d_v u_I^a - d_h \theta \quad (11)$$

$$d_v \theta = \omega. \quad (12)$$

Moreover, the equivalence class $[\mathcal{L}] \in H^{n,0}(d_h)$ determines the section EL and the equivalence class $[\omega] \in \Omega^{n-1,2}(F)/d_h \Omega^{n-2,2}(F)$ uniquely.

The following lemma is an easy consequence of the definition of ω .

Lemma 2. *When pulled back along the inclusion $\iota: \mathcal{E}_{\text{EL}} \subseteq J^{2k}F$, the image $\iota^* \omega \in \Omega_{\mathcal{E}_{\text{EL}}}^{n-1,2}(F)$ of the form ω Equation (10) is both horizontally and vertically closed:*

$$d_h^{\mathcal{E}_{\text{EL}}} \iota^* \omega = 0, \quad (13)$$

$$d_v^{\mathcal{E}_{\text{EL}}} \iota^* \omega = 0. \quad (14)$$

Proof. The horizontal and vertical differentials on \mathcal{E}_{EL} are defined by pullback along ι , that is, $d_{\text{h}}^{\mathcal{E}_{\text{EL}}} \iota^* = \iota^* d_{\text{h}}$ and $d_{\text{v}}^{\mathcal{E}_{\text{EL}}} \iota^* = \iota^* d_{\text{v}}$. Since $\omega = d_{\text{v}} \theta$ is already vertically closed in $\Omega^{n-1,2}(F)$, it is a fortiori vertically closed in $\Omega_{\mathcal{E}_{\text{EL}}}^{n-1,2}(F)$. The rest is a consequence of the nilpotence and anti-commutativity of d_{h} and d_{v} :

$$0 = d_{\text{v}}^2 \mathcal{L} = d_{\text{v}} \text{EL}_a \wedge d_{\text{v}} u^a - d_{\text{v}} d_{\text{h}} \theta, \quad (15)$$

$$d_{\text{h}} \omega = d_{\text{h}} d_{\text{v}} \theta = -d_{\text{v}} d_{\text{h}} \theta = -d_{\text{v}} \text{EL}_a \wedge d_{\text{v}} u^a, \quad (16)$$

$$d_{\text{h}}^{\mathcal{E}_{\text{EL}}} \iota^* \omega = \iota^* d_{\text{h}} \omega = -\iota^* d_{\text{v}} \text{EL}_a \wedge d_{\text{v}} u^a = 0, \quad (17)$$

where the last equality holds because EL_a and $d_{\text{v}} \text{EL}_a$ generate the ideal in $\Omega^{*,*}(F)$ annihilated by the pullback ι^* . \square

In fact, we will promote the name *presymplectic current density* to any form satisfying these properties.

Definition 1. Given a PDE system $\iota: \mathcal{E} \subset J^k F$ we call a form $\hat{\omega} \in \Omega_{\mathcal{E}}^{n-1,2}(F)$ a *presymplectic current density compatible with \mathcal{E}* if it is both horizontally and vertically closed,

$$d_{\text{h}}^{\mathcal{E}} \hat{\omega} = 0, \quad (18)$$

$$d_{\text{v}}^{\mathcal{E}} \hat{\omega} = 0. \quad (19)$$

In other words, ω represents a cocycle (that is, a vertically closed element) in the cohomology complex $(H_{\mathcal{E}}^{n-1,*}(d_{\text{h}}), d_{\text{v}})$. (Note that we are not bounding the jet order of $\hat{\omega}$.)

The particular form $\iota^* \omega$ defined by Eq. (10) will be referred to as the presymplectic current density associated to or obtained from the Lagrangian density \mathcal{L} , if there is any potential confusion.

4 Inverse problem

In the preceding section we have defined variational PDE systems showed that each one is endowed with a geometric structure (the presymplectic current). The *inverse problem of the calculus of variations* (or the *inverse problem* for short) is, given a bundle $F \rightarrow M$ and a PDE system $\iota: \mathcal{E} \subset J^k F$ of order k , to decide when it is variational.

The simpler *non-multiplier* version of the inverse problem presupposes that we are given an equation form (f, \tilde{F}^*) for \mathcal{E} . It consists of deciding whether there exists a Lagrangian density whose Euler-Lagrange equations (EL, \tilde{F}^*) are equal to (f, \tilde{F}^*) . The necessary and sufficient conditions for the non-multiplier inverse problem are known and are called the *Helmholtz conditions*. They can be formulated elegantly as the requirement that the form $f_a \wedge d_{\text{v}} u^a \in \Omega^{n,1}(F)$ be closed in a slightly extended version of the variational bicomplex [3, Ch.5], [13].

The harder *multiplier* inverse problem consists of deciding variability directly from the sub-bundle $\iota: \mathcal{E} \subset J^k F$ itself or, equivalently, *any* regular equation form (f, E) of \mathcal{E} . The name comes from the possibility of reducing it to the simpler problem by finding the right set of “multipliers” ϵ (which could also be differential operators) such that $(\epsilon \circ f, \tilde{F}^*)$ satisfies the Helmholtz conditions. Unfortunately, the multiplier inverse problem does not yet have a satisfactory solution in full generality [2, 21].

An important contribution to the subject was made in [15], as discussed in the Introduction. Henneaux showed that, for second order ODE systems that can be put into the canonical form

$$\ddot{q} - f(t, q, \dot{q}) = 0, \quad (20)$$

the existence of a symplectic form $\hat{\omega}(t, q, \dot{q})$, defined on the bundle of initial data (q, \dot{q}) over the time axis, that is conserved by the flow of the vector field associated to the ODE system (20),

$$\partial_t \hat{\omega} - \mathcal{L}_f \hat{\omega} = 0, \quad (21)$$

is equivalent to this ODE system being variational with a unique Lagrangian density \mathcal{L} (up to addition of boundary terms) whose associated symplectic form is equal to $\hat{\omega}$. Since the EL equations of \mathcal{L} will in general not be directly in the canonical form (20), this result shows that a conserved symplectic form $\hat{\omega}$ is a certificate of a positive solution of the multiplier inverse problem. Henneaux’s proof even provides a procedure to construct \mathcal{L} from $\hat{\omega}$ and the ODE system. The multiplier inverse problem, is then reduced to identifying conserved symplectic forms, which could be attacked by algebraic means.

Unfortunately, until rather recently, it has not been clear how to generalize Henneaux’s reformulation of the multiplier inverse problem to PDEs. Several aspects of the discussion in the previous paragraph are specific to ODEs: (a) the possibility of a simple canonical form like (20), (b) the geometric formulation of the ODE as a vector field, (c) a local symplectic form $\hat{\omega}$, (d) the conservation condition (21). In this section, we present a partial generalization of Henneaux’s result to PDE systems. The PDE analogs of the key aspects are (b) the geometric formulation in terms of jet bundles (as in Section 2), (c) the local covariant presymplectic current density $\hat{\omega}$ (as in Definition 1), and (d) the closure condition $d_h^\mathcal{E} \hat{\omega} = 0$. Unfortunately, we have not been able to identify simple, local analogs of the canonical form (20) and nondegeneracy of $\hat{\omega}$ (that is, being symplectic rather than just presymplectic). Due to the last caveat, the procedure given below does not produce a unique class of equivalent Lagrangian densities \mathcal{L} associated to a given PDE system \mathcal{E} and a presymplectic current $\hat{\omega}$. On the other hand, each Lagrangian density produced is in a certain sense a subsystem of \mathcal{E} : any solution of \mathcal{E} also solves the corresponding EL equations. Section 5 is an attempt to characterize the class of Lagrangian densities that can be so produced.

The key observation that connects a local presymplectic current density with a variational formulation was made in [9, Sec.4], which is a more geometric

formulation of an earlier observation made in [16]. However, these authors did not attempt to place this result in the context of other work on the inverse problem of the calculus of variations and did not remark the similarity with the previous work of Henneaux. Moreover, their calculations remained “on-shell”, which avoided lifting the Lagrangian density “off-shell” (see the proof below), which is really necessary for a solution of the inverse problem. Below, we clarify this observation and show in detail how a Lagrangian density can be constructed using a method related to cohomological descent [7] (see also Refs. [88,89] and [191] therein).

As before, consider a vector bundle $F \rightarrow M$ over an n -dimensional manifold M and a regular PDE system $\iota: \mathcal{E} \subset J^k F$ of order k . Recall also that $\bar{F}^* = \Lambda^n M \otimes_M F^*$ is the densitized dual vector bundle of F . Finally, an important hypothesis currently assumed is that the de Rham cohomology groups of \mathcal{E}^∞ and $J^\infty F$ all vanish. All other relevant notions and notation are defined in Section 2.

Theorem 3. *If there exists a presymplectic current density $\hat{\omega} \in \Omega_{\mathcal{E}}^{n-1,2}(F)$ compatible with \mathcal{E} , then there exists a local Lagrangian density $\mathcal{L} \in \Omega^{n,0}(F)$ such that the associated presymplectic current density ω coincides with $\hat{\omega}$ on solutions ($i_\infty^* \omega = p_\infty^* \hat{\omega}$) and the Euler-Lagrange PDE system \mathcal{E}_{EL} , with equation form (EL, \bar{F}^*) , is compatible with \mathcal{E} ($\mathcal{E}^\infty \subseteq \mathcal{E}_{\text{EL}}^\infty$ or all solutions of \mathcal{E} also solve \mathcal{E}_{EL}).*

Proof. The fact that $\hat{\omega}$, from the definition of a presymplectic current density, is both horizontally and vertically closed as an element of $\Omega_{\mathcal{E}}^{n-1,2}(F)$ can be easily seen to be equivalent to $\hat{\omega}$ being de Rham closed as an element of $\Omega_{\mathcal{E}^{n+1}}(F)$. Equivalently, if ω is of homogeneous degrees $(n-1, 2)$ and de Rham closed, it follows that it is both horizontally and vertically closed.

By assumption, the de Rham cohomology group $H_{\mathcal{E}^{n+1}}(F)$ is trivial. If it were not, there could be global topological obstructions to this construction, which we do not discuss in the current treatment. Then there must exist an element $\hat{\rho} \in \Omega_{\mathcal{E}}^n(F)$ such that $d^{\mathcal{E}} \hat{\rho} = \hat{\omega}$. If we expand $\hat{\rho}$ in components of homogeneous vertical and horizontal degrees, we can represent it as $\hat{\rho} = \sum_{h+v=n} \hat{\rho}_{h,v}$. This sum is finite, since $0 \leq h \leq n$ and $0 \leq v$. The equation

$$d^{\mathcal{E}} \hat{\rho} = d_v^{\mathcal{E}} \hat{\rho} + d_h^{\mathcal{E}} \hat{\rho} = \hat{\omega} \quad (22)$$

then naturally expands into the following system for the homogeneous components

$$d_v^{\mathcal{E}} \hat{\rho}_{0,n} = 0, \quad (23)$$

$$d_v^{\mathcal{E}} \hat{\rho}_{1,n-1} = -d_h^{\mathcal{E}} \hat{\rho}_{0,n}, \quad (24)$$

$$\vdots \quad (25)$$

$$d_v^{\mathcal{E}} \hat{\rho}_{n-1,1} = -d_h^{\mathcal{E}} \hat{\rho}_{n-2,2} + \hat{\omega}, \quad (26)$$

$$d_v^{\mathcal{E}} \hat{\rho}_{n,0} = -d_h^{\mathcal{E}} \hat{\rho}_{n-1,1}. \quad (27)$$

Note that $\hat{\rho}$ could be constructed by solving the above equations term by term. This method is a special case of the *cohomological descent method* [7].

As written, these equations hold “on-shell,” that is, on the ∞ -prolonged PDE manifold \mathcal{E}^∞ . But the descent equations lift “off-shell,” to the total space of jet bundle $J^\infty F$ containing \mathcal{E}^∞ . Denote the lifted forms by removing hats, $\hat{\rho}_{h,v} = \iota_\infty^* \rho_{h,v}$, with the exception $\hat{\omega} = \iota_\infty^* \omega'$. These lifts are not unique, as we could always change them by adding terms that are annihilated by the pullback to \mathcal{E}^∞ . Recall that \mathcal{E}^∞ is defined by the equations $p^\infty f = 0$, so given local coordinates (x^i, v_A) on the equation bundle E , which extend to (x^i, v_{IA}) on $J^\infty E$, the terms annihilated by the pullback to \mathcal{E}^∞ must be proportional to f_{IA} or the exterior vertical derivatives $d_v f_{IA}$. After the lift, the above equations for $\rho_{h,v}$ also only hold up to terms proportional to f_{IA} or $d_v f_{IA}$,

$$\begin{aligned} d_v \rho_{0,n} &= 0 & + f_{IA} \lambda_{0,n+1}^{IA} + d_v f_{IA} \wedge \mu_{0,n}^{IA}, \\ d_v \rho_{1,n-1} &= -d_h \rho_{0,n} & + f_{IA} \lambda_{1,n}^{IA} + d_v f_{IA} \wedge \mu_{1,n-1}^{IA}, \\ &\vdots \\ d_v \rho_{n-2,2} &= -d_h \rho_{n-3,3} + f_{IA} \lambda_{n-2,3}^{IA} + d_v f_{IA} \wedge \mu_{n-2,2}^{IA}, \end{aligned}$$

and

$$\begin{aligned} d_v \rho_{n-1,1} &= -d_h \rho_{n-2,2} + f_{IA} \lambda_{n-1,2}^{IA} + d_v f_{IA} \wedge \mu_{n-1,1}^{IA} + \omega', \\ d_v \rho_{n,0} &= -d_h \rho_{n-1,1} + f_{IA} \lambda_{n,1}^{IA} + d_v f_{IA} \wedge \mu_{n,0}^{IA}. \end{aligned}$$

Note that we have introduced the coefficient forms $\lambda_{h,v}^{IA}$ and $\mu_{h,v}^{IA}$ to parametrize the terms annihilated by the pullback to \mathcal{E}^∞ ; they are of homogeneous horizontal and vertical degrees, as indicated by their subscripts. These coefficient forms are not simply arbitrary. As shown below, $\lambda_{n,1}^{IA}$ and $\mu_{n,0}^{IA}$ will contain in them the information about the multipliers needed to solve the inverse problem.

Given local coordinates (x^i, u_I^a) on $J^\infty F$, the goal now is to construct the forms \mathcal{L} , θ , ω and $EL_a \wedge d_v u^a$, of respective degrees $(n, 0)$, $(n-1, 1)$, $(n-1, 2)$ and $(n, 1)$, such that the corresponding equations of the covariant phase space method hold:

$$d_v \mathcal{L} = EL_a \wedge d_v u^a - d_h \theta, \quad (28)$$

$$d_v \theta = \omega. \quad (29)$$

We rewrite the $(n, 0)$ -descent equation as

$$d_v(\rho_{n,0} - f_{IA} \mu_{n,0}^{IA}) = f_{IA}(\lambda_{n,1}^{IA} - d_v \mu_{n,0}^{IA}) - d_h \rho_{n-1,1} \quad (30)$$

$$= f_{IA} \epsilon_a^{IA} \wedge d_v u^a - d_h(\rho_{n-1,1} + f_{IA} \lambda_{n-1,1}^{IA}), \quad (31)$$

where integration by parts was used to construct $\lambda_{n-1,1}^{IA}$ and ϵ_a^{IA} , with the latter being $(n, 0)$ -forms. We add $d_v(f_{IA} \lambda_{n-1,1}^{IA})$ to both sides of the $(n-1, 1)$ -descent

equation and rewrite it as

$$\begin{aligned} d_v(\rho_{n-1,1} + f_{IA}\lambda_{n-1,1}^{IA}) &= \omega' - d_h\rho_{n-2,2} + f_{IA}(\lambda_{n-1,2}^{IA} + d_v\lambda_{n-1,1}^{IA}) \\ &\quad + d_v f_{IA} \wedge (\mu_{n-1,1}^{IA} + \lambda_{n-1,1}^{IA}). \end{aligned} \quad (32)$$

It is now clear that these equations take the desired form with the following identifications:

$$\mathcal{L} = \rho_{n,0} - f_{IA}\mu_{n,0}^{IA}, \quad (33)$$

$$\theta = \rho_{n-1,1} + f_{IA}\lambda_{n-1,1}^{IA}, \quad (34)$$

$$\omega = \omega' - d_h\rho_{n-2,2} \quad (35)$$

$$\begin{aligned} &\quad + f_{IA}(\lambda_{n-1,2}^{IA} + d_v\lambda_{n-1,1}^{IA}) + d_v f_{IA} \wedge (\mu_{n-1,1}^{IA} + \lambda_{n-1,1}^{IA}), \\ \text{EL}_a \wedge d_v u^a &= f_{IA}\epsilon_a^{IA} d_v u^a. \end{aligned} \quad (36)$$

The form $\text{EL}_a \wedge d_v u^a$ naturally correspond to a bundle morphism $\text{EL}: J^l F \rightarrow \tilde{F}^*$, for some finite jet degree l , which defines the Euler-Lagrange PDE system $\mathcal{E}_{\text{EL}} \subset J^l F$ via the equation form (EL, \tilde{F}^*) . From the last equation, it is obvious that the constructed Euler-Lagrange PDE system contains the original one, $\mathcal{E}^\infty \subseteq \mathcal{E}_{\text{EL}}$. \square

Note that all the above calculations were done on jet bundles of infinite order. However, each of the forms introduced at intermediate steps depends only on jet coordinates of some finite order. Therefore, the final Lagrangian will also depend on jet coordinates up to some finite order, which may be much higher than the order of the original PDE system. Note that the degree of the Lagrangian could be artificially inflated by the presence of boundary terms like $d_h \mathcal{B}$, where \mathcal{B} could depend on jet coordinates of some high order.

It is also important to remark that the resulting Euler-Lagrange equations may not be equivalent to the full original PDE system, but only to a subsystem thereof or a “weaker” system, one whose solution space contains all solutions of \mathcal{E} , but may be strictly larger. This is unavoidable, since the PDE system may consist, for example, of several uncoupled subsystems, each of which may have an independent variational formulation. More complicated situations are of course possible. In the ODE context, the requirements that $\hat{\omega}$ actually be symplectic (rather than just presymplectic) and that the ODE system is in the canonical form (20) are sufficient to guarantee that the original system is fully variational [15]. Unfortunately, at least one of these conditions fails already when the ODE system is not determined (possibly because of gauge invariance) or under the inclusion of algebraic equations that constrain the initial data (q, \dot{q}) . In the PDE case, in analogy with the ODE one, there may be a set of conditions on the symbol of the PDE system and on $\hat{\omega}$ that guarantees a fully variational formulation, which may be further complicated by allowing equations with gauge symmetries or constraints. However, such analogous conditions remain to be investigated in detail.

5 Arbitrariness in Lagrangian density construction

There were a number of choices involved in the construction of the (off-shell) Lagrangian density \mathcal{L} from the (on-shell) presymplectic current density $\hat{\omega}$. In this section, we investigate how the resulting \mathcal{L} depends on these choices.

The choices are exhausted by the following substitutions:

$$(i) \quad \hat{\omega} \rightarrow \hat{\omega} + d_h^\mathcal{E} \hat{\pi}_{n-2,2}, \quad \text{with} \quad d_v^\mathcal{E} \hat{\pi}_{n-2,2} = 0; \quad (37)$$

$$(ii) \quad \hat{\rho} \rightarrow \hat{\rho} + d^\mathcal{E} \hat{\sigma}; \quad (38)$$

$$(iii) \quad \rho \rightarrow \rho + f_{IA} \bar{\lambda}^{IA} + d_v f_{IA} \wedge \bar{\mu}^{IA}, \quad (39)$$

$$\omega' \rightarrow \omega' + f_{IA} \bar{\lambda}'^{IA} + d_v f_{IA} \wedge \bar{\mu}'^{IA}. \quad (40)$$

We deal with each kind of substitution one by one.

(i) It is easy to see the following identity:

$$d^\mathcal{E}(\hat{\rho} + \hat{\pi}_{n-2,2}) = \hat{\omega} + d_h^\mathcal{E} \hat{\pi}_{n-2,2}. \quad (41)$$

Since the change $\hat{\rho} \rightarrow \hat{\rho} + \hat{\pi}_{n-2,2}$ affects neither of the $\hat{\rho}_{n,0}$ or $\hat{\rho}_{n-1,1}$ components, the off-shell lift of the $(n,0)$ descent equation is unmodified. Therefore, the Lagrangian density \mathcal{L} does not change.

(ii) Lifting $\hat{\sigma}$ off-shell to σ and using the decomposition into homogeneous components, $\sigma = \sum_{h+v=n} \sigma_{h,v}$, we see the change

$$\rho_{h,v} \rightarrow \rho_{h,v} + d_h \sigma_{h-1,v} + d_v \sigma_{h,v-1}. \quad (42)$$

However, since $d(\rho + d\sigma) = d\rho$, the descent equations are unmodified. In the end, the Lagrangian changes only as $\mathcal{L} \rightarrow \mathcal{L} + d_h \sigma_{n-1,0}$. Since the change is by a horizontally exact term, the EL equations remain the same.

(iii) This kind of substitution does in general change the equivalence class of the Lagrangian density, that is, the EL equations of the modified Lagrangian density may be different. The forms parametrizing the failure of the off-shell lift of the descent equations undergo the change (except for the $\lambda_{n-1,2}$ and $\mu_{n-1,1}$ coefficients, which undergo an obvious additional changes compensating the change in ω' , which ultimately does not affect \mathcal{L})

$$\lambda_{h,v}^{iIA} \rightarrow \lambda_{h,v}^{iIA} + (d_h \bar{\lambda}_{h-1,v}^{iIA} + dx^{(i} \wedge \bar{\lambda}_{h-1,v}^{I)A}) + d_v \bar{\lambda}_{h,v-1}^{iIA}, \quad (43)$$

$$\mu_{h,v}^{iIA} \rightarrow \mu_{h,v}^{iIA} - (d_h \bar{\mu}_{h-1,v}^{iIA} + dx^{(i} \wedge \bar{\mu}_{h-1,v}^{I)A}) + (\bar{\lambda}_{h,v}^{iIA} - d_v \bar{\mu}_{h,v-1}^{iIA}). \quad (44)$$

Therefore, the Lagrangian density undergoes the change

$$\mathcal{L} \rightarrow \mathcal{L} + \bar{\mathcal{L}} = \mathcal{L} + f_{iIA} \bar{\lambda}_{n,0}^{iIA} - f_{iIA} [\bar{\lambda}_{n,0}^{iIA} - (d_h \bar{\mu}_{n-1,0}^{iIA} + dx^{(i} \wedge \bar{\mu}_{n-1,0}^{I)A})] \quad (45)$$

$$= \mathcal{L} + f_{iIA} (d_h \bar{\mu}_{n-1,0}^{iIA} + dx^{(i} \wedge \bar{\mu}_{n-1,0}^{I)A}). \quad (46)$$

In general, the multipliers ϵ_a^{IA} will change as well, say $\epsilon_a^{IA} \rightarrow \epsilon_a^{IA} + \bar{\epsilon}_a^{IA}$. Thus, the EL equations of \mathcal{L} and $\mathcal{L} + \bar{\mathcal{L}}$ may not be equivalent. In other words, the

equivalence classes $[\mathcal{L}]$ and $[\mathcal{L} + \tilde{\mathcal{L}}]$ will differ. Though both sets of EL equations will be consequences of \mathcal{E} .

At this point, it is worth reflecting on when two Lagrangians \mathcal{L} and \mathcal{L}' should be considered equivalent. The standard answer is iff they differ by a boundary term, $\mathcal{L}' - \mathcal{L} = d_h \mathcal{B}$. However, consider the simple 1-dimensional Lagrangians

$$\mathcal{L}[q_1, q_2, \lambda] = \left[\frac{1}{2} \dot{q}_1^2 + \frac{1}{2} \dot{q}_2^2 + \lambda(q_1 - q_2) \right] dt, \quad (47)$$

$$\mathcal{L}'[q_1, q_2, \lambda] = \left[\frac{1}{2} \dot{q}_1^2 + \frac{1}{2} \dot{q}_2^2 + (\lambda + \alpha)(q_1 - q_2) \right] dt, \quad (48)$$

where α is a constant (though in principle it could be a more complicated function of λ and q_i). Note that the difference between the two Lagrangians, $\mathcal{L}' - \mathcal{L} = \alpha(q_1 - q_2)$, is not a boundary term, though it is proportional to the constraint equation $q_2 - q_1 = 0$ obtained by varying λ . It is easy to check that the resulting Euler-Lagrange equations for either Lagrangian are equivalent to the set $\ddot{q}_1 = 0, q_2 = q_1, \lambda = 0$. Moreover, their symplectic currents agree as well, $\omega = \sum_i d_v \dot{q}_i \wedge d_v q_i = \omega'$. Furthermore, since we never limited the order of the Lagrangian density, it could be of order zero (a ordinary, non-differential variational problem). Then both $\omega = \omega' = 0$ and the critical points of \mathcal{L} and \mathcal{L}' coincide as long as $d_v(\mathcal{L}' - \mathcal{L})$ vanishes at all critical points. Therefore, in the presence of constraints on the initial data, the classes of equivalent Lagrangian densities are larger than just those that differ by boundary terms, if equivalence is evaluated in terms of the Euler-Lagrange equations and the presymplectic current.

Consider the following preorder² on Lagrangian densities. We say that \mathcal{L}' contains \mathcal{L} , $\mathcal{L} \prec \mathcal{L}'$, if the ∞ -jet bundle submanifolds defined by the prolongations of the corresponding Euler-Lagrange equations obey $\mathcal{E}_{\text{EL}}^\infty \subseteq \mathcal{E}_{\text{EL}'}^\infty$ and their presymplectic current densities agree up to horizontally exact term when pulled back to the more restrictive equation, $\iota_\infty^* \omega = \iota_\infty^* \omega'$, where $\iota: \mathcal{E}_{\text{EL}} \subset J^\infty F$. The relation \prec is clearly transitive and reflexive and hence a preorder.

The containment relation is however not a partial order, because it is not antisymmetric. As already discussed earlier, if $\mathcal{L}' - \mathcal{L} = d_h \mathcal{B}$, then $\mathcal{L} \prec \mathcal{L}'$ and $\mathcal{L}' \prec \mathcal{L}$, while $\mathcal{L} \neq \mathcal{L}'$. On the other hand, this preorder gives rise to the equivalence relation $\mathcal{L} \sim \mathcal{L}'$ (iff $\mathcal{L} \prec \mathcal{L}'$ and $\mathcal{L}' \prec \mathcal{L}$) and projects to an actual partial order $[\mathcal{L}] \prec [\mathcal{L}']$ on the equivalence classes with respect to \sim .

Given a PDE system $\iota: \mathcal{E} \subset J^k F$ with a compatible presymplectic current density $\hat{\omega}$, define the set $L_{\mathcal{E}, \hat{\omega}}$ to consist of equivalence classes of Lagrangian densities $[\mathcal{L}]$ such that $\mathcal{E}^\infty \subseteq \mathcal{E}_{\text{EL}}^\infty$ and $\iota_\infty^* \omega = p_\infty^* \hat{\omega}$. Then $L_{\mathcal{E}, \hat{\omega}}$ is an upper set with respect to the \prec partial order (which means that $[\mathcal{L}] \in L_{\mathcal{E}, \hat{\omega}}$ and $[\mathcal{L}] \prec [\mathcal{L}']$ implies $[\mathcal{L}'] \in L_{\mathcal{E}, \hat{\omega}}$). After all, as should be clear from the definition of the \prec

²A *preorder* P on a set X is a relation such that is *reflexive* (xPx) and *transitive* (xPy and yPz implies xPz), but in general neither symmetric nor antisymmetric. All partial orders and equivalence relations are preorders. The maximum symmetric subrelation E (xEy iff xPy and yPx) is an equivalence relation and the quotient $X \rightarrow X/E$ projects P to a partial order P/E on X/E (which is necessarily antisymmetric) [22].

relation, if $\rho = \mathcal{L} + \theta$ could arise in the construction in the preceding section (that is $\mathcal{L} \in L_{\mathcal{E}, \hat{\omega}}$), then so could $\rho = \mathcal{L}' + \theta'$ (that is $\mathcal{L}' \in L_{\mathcal{E}, \hat{\omega}}$) for any \mathcal{L}' that contains \mathcal{L} ($\mathcal{L} \prec \mathcal{L}'$).

It is clear that the construction of the preceding section can yield any element of $L_{\mathcal{E}, \hat{\omega}}$ as a partial solution of the inverse problem. It is possible that a consideration of the structure of the partial order \prec as well as the decomposability and minimal elements of $L_{\mathcal{E}, \hat{\omega}}$ in relation with the symbol of the PDE system \mathcal{E}^∞ can yield a definite solution of the inverse problem. These questions are yet to be investigated in detail.

6 Conclusion

We have shown, following a recent observation in [9, 16] and a strong analogy with previous work in [15, 17, 1], that a horizontally conserved presymplectic current density is a certificate that a subsystem of a given PDE system or a comparatively weaker system is variational (any solution of the original PDE system also solves the obtained variational system). If this subsystem is actually the full system, then the multiplier inverse problem of the calculus of variations has a positive solution.

Restricting to the context of second order ODEs that can be put in canonical form (20), it is known that any two Lagrangians whose Euler-Lagrange equations are equivalent to the given ODE system \mathcal{E} and that have equivalent symplectic current densities $\hat{\omega}$ must differ by a boundary term[15]. In that case, the equivalence class of Lagrangians solving the full inverse problem of the calculus of variations for \mathcal{E} depends only on the characteristic cohomology class in $[\hat{\omega}] \in H_{\mathcal{E}}^{n-1,2}(\mathrm{d}_h)$. However, in more general contexts (for general PDEs, or even ODEs with constraints) a characterization of equivalent Lagrangians (those sharing the same set of solutions and equivalent presymplectic current densities) is still missing.

On the other hand, in Section 5 we defined a preorder relation \prec on Lagrangian densities $\mathcal{L} \in \Omega^{n,0}(F)$ that relates both this equivalence problem for Lagrangian densities and the ambiguity in the Lagrangians produced by the construction of Section 4. The precise structure of the preorder \prec and the conditions on a PDE system and compatible presymplectic current that guarantee that it is fully variational (rather than just a subsystem thereof) remain to be investigated. It is possible that generic methods for determining the characteristic cohomology groups of a PDE [10, 23] would be helpful in classifying possible presymplectic current densities and hence variational formulations.

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